ON A THEOREM IN GENERAL ANALYSIS AND THE INTERRELATIONS OF EIGHT FUNDAMENTAL PROPERTIES OF CLASSES OF FUNCTIONS.*

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INTRODUCTION.

In his "Introduction to a Form of General Analysis"† E. H. MOORE defined eight properties of classes of functions:

$$\mathrm{L, C, D, A, \Delta, K_1, K_2, K_{12*}}.‡$$

If we denote by $\mathfrak{P}$ a general class, Δ a development of $\mathfrak{P}$ (cf. § 1), $\mathfrak{M}$ a class of functions μ on $\mathfrak{P}$ to the class $\mathfrak{A}$ of real numbers; then the properties L, C, D, A are independent of the character of $\mathfrak{P}$, while the properties Δ, $\mathrm{K_1}$, $\mathrm{K_2}$, $\mathrm{K_{12*}}$ are defined in terms of the development Δ. The property $\mathrm{K_1}$ is a generalization of convergence, the property $\mathrm{K_2}$ of continuity, while the property $\mathrm{K_{12*}}$ is defined in terms of the functions which are at once convergent and continuous on $\mathfrak{P}$ relative to Δ and $\mathfrak{M}$. The eight properties are not independent but admit among others the important relation: $\mathrm{D\Delta K_1K_2}$ imply $\mathrm{K_{12*}}$.§

A. D. PITCHER‖ investigated the interrelations of these eight properties and their negatives and found that 108 combinations are impossible, and exhibited examples showing the possibility of 145. There remained three combinations whose character was left undetermined, out of a conceivable total of 256. His investigations led him to the conclusion that the property $\mathrm{K_{12*}}$ is implied by the composite property $\mathrm{D_1\Delta K_1K_2}B_3$, where $\mathrm{D_1}$ is a dominance property weaker than D, and B_3 is defined in terms of a development Δ.¶

The developments Δ assumed by MOORE and PITCHER are finite, that is, each stage consisted of a finite number of subclasses of $\mathfrak{P}$. The author

* Presented to the American Mathematical Society at Chicago, April 21, 1916.

† The New Haven Mathematical Colloquium (Yale University Press, New Haven, 1910), pp. 1–150. To avoid frequent acknowledgments the author wishes to state that the symbolism and terminology employed are due to MOORE and A. D. PITCHER (citation below).

‡ The eight properties are defined in the following pages in the order in which they occur in the discussion.

§ MOORE, loc. cit., p. 145, Theorem III.

‖ "Interrelations of Eight Fundamental Properties of Classes of Functions," *The Kansas University Science Bulletin*, Vol. VII, No. 1, (1913), pp. 1–67.

¶ PITCHER, loc. cit., p. 44, Theorem I.

removed this restriction and extended the theory of MOORE to the case of infinite developments, obtaining a theorem of which the two theorems of MOORE and PITCHER mentioned are instances.*

It is the purpose of the present paper to show that the hypothesis B_3 of PITCHER is non-essential and that K_{12*} is, in fact, implied by $D_1 \Delta K_1 K_2$. It results from this proposition that two of the three properties whose character with respect to existence was not determined are self contradictory (cf. § 7).

§ 1. DEFINITIONS AND NOTATION.—We are concerned with classes $\mathfrak{M}$ of real and single-valued functions which are assumed to be defined for each element (point) of a given general class (space) $\mathfrak{P}$, and with properties of such classes $\mathfrak{M}$ defined in terms of a development Δ of $\mathfrak{P}$. Denoting the class of all real numbers by $\mathfrak{A}$, we speak of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

A development Δ of a class $\mathfrak{P}$ is a system $((\mathfrak{P}^{ml}))$ of subclasses $\mathfrak{P}^{ml}$ of $\mathfrak{P}$; the indices m, l being integral valued and the range of the index l, for fixed m, dependent on the value of m. The system of all classes $\mathfrak{P}^{ml}$ for a fixed value of the index m forms stage m of the development. A stage of a development is finite if the corresponding system $(\mathfrak{P}^{ml})$ is *finite*. A development is *finite* in case every stage is finite.†

Unless the contrary is stated the developments which occur in the following discussion are assumed to be finite. There will be, for each value of m, a finite number L^m ($\geqq 0$) of classes $\mathfrak{P}^{ml}$ of stage m.

If a point p belongs to a class $\mathfrak{P}^{ml}$, it is said to be *developed* of stage m. The class of all such points is denoted by $\mathfrak{P}^m$. Its complement $\mathfrak{P} - \mathfrak{P}^m$ is represented by the symbol $\mathfrak{P}^{-m}$, denoting the class of all points undeveloped of stage m.

THE RELATION K_{pm}. If a point p is undeveloped of stage m, or some later stage (that is, stage of index $m' > m$), it is in the relation K_{pm} relative to a development Δ of the space $\mathfrak{P}$. The class of all points p in the relation K_{pm} for given m is denoted by $\mathfrak{P}_m$. The complement $\mathfrak{P}_{-m}$ of $\mathfrak{P}_m$ is a very important class, in fact, the class of all points developed of stage m and every succeeding stage. The statement that p belongs to $\mathfrak{P}_{-m}$ is represented by the symbol $\overline{K}_{pm}$.

THE RELATION $K_{p_1p_2m}$. Two points p_1, p_2 of $\mathfrak{P}$ are *connected* of stage m in case there is a class $\mathfrak{P}^{ml}$ which contains both. If two points p_1, p_2 are connected of stage m or some later stage, they are in the relation $K_{p_1p_2m}$.

THE DOMINANCE PROPERTY D_1. A class $\mathfrak{M}$ of functions has the domi-

* "Infinite Developments and the Composition Property $(K_{12}B_1)_*$ in General Analysis."

† A development is *ultimately finite* in case there is an integer m_0 such that for all values of $m \geqq m_0$, stage m of the development is finite. The theory here developed is concerned only with the ultimate character of the developments which enter and there is no loss of generality in the restriction to *finite* instead of *ultimately finite* developments.

nance property D_1 in case for every pair μ_1, μ_2 of functions of $\mathfrak{M}$ there is a constant a and a function μ of $\mathfrak{M}$ such that the inequalities,

$$(1) \qquad |\mu_1| \leqq a|\mu|, \qquad |\mu_2| \leqq a|\mu|,$$

hold uniformly on $\mathfrak{P}$. [It is customary, when writing inequalities between functions which hold uniformly in the variables, to suppress the variables.]

THE PROPERTY K_2. A class $\mathfrak{M}$ has the property $K_2(K_2\mathfrak{M})$ relative to to a development Δ of the range $\mathfrak{P}$ in case for every function μ of the class $\mathfrak{M}$ there is a function μ_2 of $\mathfrak{M}$ such that *for every small positive number e there is a positive integer m_e* such that whenever two points p_1, p_2 are in the relation $K_{p_1p_2m_e}$ then

$$(2) \qquad |\mu_{p_1} - \mu_{p_2}| \leqq e|\mu_{2p_1}|.$$

[Henceforth we shall use the abbreviation: *there is an m_e*, to replace the italicized part of the preceding definition. Thus the definition of K_2 might be stated: there exists μ_2 and m_e such that $K_{p_1p_2m_e}$ implies, etc.]

§ 2. SOME CONSEQUENCES OF THE PROPERTY K_2.—If we denote the oscillation of a function μ on a class $\mathfrak{P}^{ml}$ of a development Δ of the range $\mathfrak{P}$ by $\omega^{ml}(\mu)$ and the lower bound of μ on $\mathfrak{P}^{ml}$ by $B^{ml}(\mu)$, it is easy to see from inequality (2) that whenever m exceeds m_e then

$$(3) \qquad \omega^{ml}(\mu) \leqq 2eB^{ml}(\mu_2).$$

It follows that if μ_2 vanishes on $\mathfrak{P}^{ml}$ $(m \geqq m_1)$ or has the lower bound zero, then μ is constant on $\mathfrak{P}^{ml}$. Therefore, unless μ vanishes identically on $\mathfrak{P}^{ml}$, μ is bounded from zero on that class. Conversely, if μ is not constant on $\mathfrak{P}^{ml}$, then μ_2 must be bounded from zero on $\mathfrak{P}^{ml}$. Consequently, if $\bar{\mu}$ is the common dominant of μ and μ_2, the function $\bar{\mu}$ is (in view of inequality (1) and the preceding remarks) bounded from zero on every class $\mathfrak{P}^{ml}$ $(m \geqq m_1)$. This result is stated in the following theorem:

If a class $\mathfrak{M}$ of functions on a general range $\mathfrak{P}$ with a development Δ has the property K_2 relative to Δ and the property D_1, then for every function μ of the class $\mathfrak{M}$ there is a function $^{\mu}$ and an integer m_1 such that, for every $m \geqq m_1$ and class $\mathfrak{P}^{ml}$, either μ vanishes identically on $\mathfrak{P}^{ml}$ or $\bar{\mu}$ is bounded from zero on $\mathfrak{P}^{ml}$.

The classes $\mathfrak{P}^{ml}$ on which μ vanishes identically form a set which may be null but is almost denumerably infinite. Relative to the function μ, we represent by $\mathfrak{P}^{ml_0}$ the classes on which μ is identically zero and by $\mathfrak{P}^{ml_1}$ the remaining classes of the development. The system Δ_0 of all classes $\mathfrak{P}^{ml_0}$ forms a development of $\mathfrak{P}$.* Likewise the system of all classes $\mathfrak{P}^{ml_1}$ forms a development Δ_1 of $\mathfrak{P}$. In the developments Δ_0, Δ_1 it is understood that

* We shall find it convenient to say that the function μ vanishes identically on the development Δ_0.

the stages $m < m_1$ are empty. For the purposes of this paper the development Δ may be regarded as equivalent to a development $\Delta_0 + \Delta_1$.

We understand by a *representative* system $((r^{ml}))$ of a development a system of points r^{ml}, where r^{ml} is an element of the class $\mathfrak{P}^{ml}$. If a class $\mathfrak{P}^{ml}$ is null, it will not have a representative element.

Let $\mathfrak{M}$ be a class with the property K_2, and suppose that p_1, p_2 are common elements of a class $\mathfrak{P}^{ml}$ of a stage $m \geqq m_e$ (effective in inequality (2)). Then if we replace p_1 by r^{ml} and let p_2 represent any element p of $\mathfrak{P}^{ml}$, we may write

$$|\mu_p| \leqq e\,|\mu_{2r^{ml}}| + |\mu_{r^{ml}}|\,.$$

If we set $e = 1$ and let a_m be the greatest value of $|\mu_{2r^{me}}| + |\mu_{p^{me}}|$ for all points r^{ml} representative of classes of stage m, then we have for all points of $\mathfrak{P}^m$ $(m \geqq m_1)$

$$(4) \qquad |\mu| \leqq a_m.$$

It should be noted here that the value of m_1 depends on μ.

§ 3. THE DEVELOPMENTAL PROPERTY Δ.—By definition, a class $\mathfrak{M}$ with the developmental property Δ contains a system $\mathfrak{D} = ((\delta^{ml}))$ of functions subject to two conditions of which we need consider only the first at this time: $(1a)$ there is an m_e such that, for all $m \geqq m_e$,

$$(5) \qquad \Big|\sum_g{}' \delta_p^{mg} - 1\Big| \leqq e, \qquad \Big|\sum_g{}' |\delta_p^{mg}| - 1\Big| \leqq e,$$

where g denotes a value of the index l for which p belongs to $\mathfrak{P}^{ml}$, and the prime on the summation sign denotes that the summation is restricted to existent classes of stage m. The condition applies only in case p is *developed* of stage m.

Let $\mathfrak{M}$ be a class with the properties D_1, Δ. Since the development Δ is finite there are but a finite number of developmental functions δ^{ml} of each stage m. Hence from D_1, there is a function μ_m such that, for all indices l of stage m,

$$(6) \qquad |\delta_p^{ml}| \leqq |\mu_m|\,.$$

If we give e in inequalities (5) the value $\frac{1}{2}$, then for every point p developed of stage m there is at least one index l for which

$$(7) \qquad |\delta_p^{ml}| \geqq \frac{1}{2L^m},$$

where L^m is the number of classes $\mathfrak{P}^{ml}$ of stage m of Δ.

Therefore, the function μ_m is bounded from zero on $\mathfrak{P}^m$; in fact, for every point p developed of a stage m ($m \geqq m_{\frac{1}{2}}$, effective in inequalities

(5)), we have

$$|\mu_{mp}| \geqq \frac{1}{2L^m}. \tag{8}$$

THE PROPERTY K_1. A class $\mathfrak{M}$ has the property $K_1(K_1\mathfrak{M})$ relative to a development Δ in case for every function μ of $\mathfrak{M}$ there is a function μ_1 of $\mathfrak{M}$ and an m_e such that K_{pm_e} implies

$$|\mu_p| \leqq e|\mu_{1p}|. \tag{9}$$

THE RELATION K_{pmm_1}. If p is a point in the relation $K_{pp_1m_1}$ with a point p_1 of $\mathfrak{P}_{-m}$ (§ 1), then p is in the relation K_{pmm_1}. The class of all such points is the class $\mathfrak{P}_{mm_1}$. It is evident that if m_1 is not less than m, $\mathfrak{P}_{mm_1} \geqq \mathfrak{P}_{-m}$.

If the class $\mathfrak{M}$ has the composite property $D_1\Delta K_1K_2$, there is for every integer m an integer $m_1 \geqq m$ such that $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$.

Suppose that the inclusion $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$ fails for every value of $m_1 > m$. Then there must exist a sequence $\{p_m\}$ of points such that for every m_1 there is a point q_{m_1} of $\mathfrak{P}_{-m}$ in the relation $K_{p_{m_1}q_{m_1}m_1}$, while p_{m_1} is not in $\mathfrak{P}_{-m_1}$, that is, the relation $K_{p_{m_1}m_1}$ holds.

Let μ be any function of the class $\mathfrak{M}$. Then, because $\mathfrak{M}$ has the property K_2, we have μ_2 and m_e such that K_{pqm_e} implies

$$|\mu_p - \mu_q| \leqq e|\mu_{2q}|. \tag{2'}$$

Since $\mathfrak{P}_{-m}$ belongs to $\mathfrak{P}^{m_1}$ for every value of $m_1 > m$, it follows from (4) that μ and μ_2 are limited on $\mathfrak{P}_{-m}$. Therefore μ and μ_2 are limited on the sequence $\{q_m\}$. Hence μ and μ_2 are limited on the sequence $\{p_m\}$.

But, from the hypothesis that the class $\mathfrak{M}$ has the property K_1, there is a function μ_1 and an integer m_e such that K_{pm_e} implies

$$|\mu_p| \leqq e|\mu_{1p}|.$$

Since μ_1 must be bounded on the sequence $\{p_m\}$ it follows from the above inequality and the hypothesis on the sequence $\{p_m\}$ that $|\mu_{p_m}|$ tends toward zero with m.

However, the class $\mathfrak{M}$ contains a function μ bounded from zero on $\mathfrak{P}^{m_1}$ for some value of $m_1 > m$. This function μ must therefore exceed some number $a_0 > 0$ at all points of $\mathfrak{P}_{-m}$ ($\leqq \mathfrak{P}^{m_1}$). But from inequality (2′) we infer, for all k such that $k \geqq m_e$, that the relation $K_{p_kq_km_e}$ holds, and

$$\begin{aligned}|\mu_{p_k}| &\geqq |\mu_{q_k}| - e|\mu_{2q_k}| \\ &\geqq a_0 - eb_2,\end{aligned}$$

where b_2 denotes the least upper bound of μ_2 on $\mathfrak{P}_{-m}$. Since e is arbitrary, the sequence $|\mu_{p_k}|$ cannot tend towards zero, contrary to the result pre-

viously obtained. Hence the hypothesis that $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$ fails for every $m_1 > m$ is untenable and the theorem is established.

From the results of § 2 and the present article we readily infer the following:

If a class $\mathfrak{M}$ *has the composite property* $D_1\Delta K_2$, *every function* μ *of* $\mathfrak{M}$ *is bounded on* $\mathfrak{P}_{-m}$ *for every value of m. For every positive integer m there is a function* μ_m *bounded from zero on* $\mathfrak{P}_{-m}$.

Furthermore, if $\mathfrak{M}$ *has the additional property* K_1, *then, for every positive integer m, there is an integer* $m_1 > m$ *such that* $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$, *every function* μ *is bounded on* $\mathfrak{P}_{mm_1}$, *and there is a function* μ_{m_1} *bounded from zero on* $\mathfrak{P}_{mm_1}$.

§ 4. ON FUNCTIONS OF THE CLASS $(\mathfrak{M}'\mathfrak{M}'')_*$.—Denoting by $\mathfrak{P}'$, $\mathfrak{P}''$ two ranges which are conceptually distinct and by $\mathfrak{M}'$, $\mathfrak{M}''$ classes of functions μ', μ'' on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively, we define the class $(\mathfrak{M}'\mathfrak{M}'')_*$ as the class of all functions which are the limits of sequences of linear combinations of functions of the multiplicational composite class $\mathfrak{M}'\mathfrak{M}''$, the convergence being uniform relative to a scale function from the class $\mathfrak{M}'\mathfrak{M}''$. That is, if we introduce the notation

$$\Theta_n = \sum_{j=1}^{j=n} a_{nj}\mu'_{nj}\mu''_{nj},$$

the class $(\mathfrak{M}'\mathfrak{M}'')_*$ is the class of all functions Θ of the form

$$\Theta = L_n\Theta_n,$$

where the convergence is uniform on $\mathfrak{P}'\mathfrak{P}''$ relative to some scale function $\mu'\mu''$ of the class $\mathfrak{M}'\mathfrak{M}''$. By definition, there is an n_e such that for all $n \geqq n_e$

$$|\Theta - \Theta_n| \leqq e|\mu'\mu''|,$$

uniformly on $\mathfrak{P}'\mathfrak{P}''$.

The following proposition is an immediate consequence of this inequality:

If the classes $\mathfrak{M}'$, $\mathfrak{M}''$ *each have the dominance property* D_1, *there exist a function* $\tau_1 = \tau'_1\tau''_1$ *of* $\mathfrak{M}'\mathfrak{M}''$ *and an integer* n_1 *such that, for all* $n \geqq n_1$, $a\tau_1$ *dominates* Θ, $\mu'\mu''$, *and* Θ_n *whenever a is a sufficiently large positive constant.*

Let $\Delta'_0 = ((\mathfrak{P}'^{ml_0}))$ be the system of all classes $\mathfrak{P}'^{ml_0}$ on which τ'_1 vanishes identically. Then all the functions Θ, $\mu'\mu''$, Θ_n $(n \geqq n_1)$ vanish identically on Δ_0. We recall, from § 2, that there exists a function σ' which dominates τ'_1 and is bounded from zero on each class $\mathfrak{P}'^{ml_1}$ of the development Δ'_1 whenever the class $\mathfrak{M}'$ has the properties D_1, K'_2.

The following theorem of PITCHER (loc. cit., p. 30) is of importance in the sequel:

If relative to a development Δ' *of* $\mathfrak{P}'$ *the class* $\mathfrak{M}'$ *has the property* K'_1, *and if* $\mathfrak{M}'$ *and* $\mathfrak{M}''$ *each have the dominance property* D_1, *then every function* Θ *of* $(\mathfrak{M}'\mathfrak{M}'')_*$ *has the property* $K_1\mathfrak{M}'(\mathfrak{M}'')$.

§ 5. CONDITIONS IMPLYING THAT THE CLASS $(\mathfrak{M}'\mathfrak{M}'')_*$ HAVE THE PROPERTY $K_2'\mathfrak{M}'(\mathfrak{M}'')$.—MOORE has shown that if $\mathfrak{M}'$ has the property DK_{12}', and $\mathfrak{M}''$ has the property D,* the class $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$.† PITCHER showed that the property D in the hypothesis on $\mathfrak{M}'$, can be replaced by the weaker property D_1B_3'. The property B_3' will be defined later. We shall show that the conclusion is true under the hypothesis, $\mathfrak{M}'$ has the property $D_1\Delta'K_1'K_2'$.

If a class $\mathfrak{M}'$ has the composite property $D_1\Delta'K_{12}'$, and a class $\mathfrak{M}''$ has the property D, every function of the class $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$.

The proof of this theorem follows the plan of the proof of the corresponding theorem of PITCHER,‡ differing from his proof in details whose character will be indicated later.

From the theorem of MOORE stated in § 4 there exists for every function Θ of $(\mathfrak{M}'\mathfrak{M}'')_*$ a scale function $\mu_1'\mu_1''$ such that $K_{pm_e'}'$ implies:

$$|\Theta_{p'}| \leqq e|\mu_{1p'}'\mu_1''|. \tag{10}$$

Since $\mathfrak{M}'$ has the property K_2' there is a function μ_0' and integer m_e'' such that $K_{p_1'p_2'm_e''}'$ implies:

$$|\mu_{1p_1'}' - \mu_{1p_2'}'| \leqq e|\mu_{0p_1'}'|. \tag{11}$$

It readily follows from the two preceding inequalities that there is an m_e such that the simultaneous presence of the relations

$$K_{p_1'm_e}', \qquad K_{p_2'm_e}', \qquad K_{p_1'p_2'm_e}'$$

implies

$$|\Theta_{p_1'}| + |\Theta_{p_2'}| \leqq e|\mu_{0p_1'}'\mu_1''|. \tag{12}$$

By definition, if a class $\mathfrak{M}$ has the dominance property D, there exists for every sequence $\{\mu_m\}$ of functions of $\mathfrak{M}$ a function μ and sequence $\{a_m\}$ of positive real numbers such that for every m, $|a_m\mu| \geqq |\mu_m|$. It is at once evident that any class of functions which is limited on a range $\mathfrak{Q}$ and contains a function bounded from zero on $\mathfrak{Q}$ has the property D.

The development Δ' of $\mathfrak{P}'$ determines by an obvious reduction a development Δ_m' of the class $\mathfrak{P}_{-m}'$. From the results of §§ 2, 3 it follows that

* A class $\mathfrak{M}$ has the dominance property D in case there exists for every sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ a function μ of $\mathfrak{M}$ and sequence $\{a_n\}$ of positive constants such that for every n, $|\mu_n| \leqq a_n|\mu|$.

† A function θ on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$ in case there is a function $\mu'\mu''$ of $\mathfrak{M}'\mathfrak{M}''$ and an m_e such that $K'_{p_1'p_2'm_e}$ implies

$$|\Theta_{p_1'} - \Theta_{p_2'}| \leqq e|\mu_{p_1'}'\mu''|.$$

A class N of functions on $\mathfrak{P}'\mathfrak{P}''$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$ in case every function Θ of N has the property.

‡ Cf. PITCHER, loc. cit., § 39, pp. 40–43.

relative to $\mathfrak{P}'_{-m}$ the class $\mathfrak{M}'$ has the property D. It is easy to see that every function has the property K_1' relative to the development Δ_m' of $\mathfrak{P}'_{-m}$, since no element of $\mathfrak{P}'_{-m}$ is undeveloped after stage m. Hence the class $\mathfrak{M}_m'$ of functions on $\mathfrak{P}'_{-m}$, obtained by considering the functions of $\mathfrak{M}$ on $\mathfrak{P}'_{-m}$ only, has the composite property DK_{12}' relative to to Δ_m'. The result of MOORE cited above applies here, and we conclude that relative to Δ_m' the function Θ has the property $\mathrm{K}_2'\mathfrak{M}'(\mathfrak{M}'')$. That is, there exists, for every integer m, a function $\mu_m'\mu_m''$ of $\mathfrak{M}'\mathfrak{M}''$ and an integer m_{em} such that $\mathrm{K}_{p_1'p_2'm_{em}}'$ implies (when p_1', p_2' are points of P'_{-m})

$$(13) \qquad |\Theta_{p_1'} - \Theta_{p_2'}| \leqq e\,|\mu'_{mp_1}\mu_m''|.$$

We have seen in § 4 that there exist in $\mathfrak{M}'\mathfrak{M}''$ a function $\tau_1 = \tau_1'\tau_1''$ such that τ_1 dominates Θ, and that, relative to τ_1', $\Delta' = \Delta_0' + \Delta_1'$, where τ_1' vanishes identically on Δ_0', and a function σ', dominating τ_1', which is bounded from zero on every class $\mathfrak{P}^{ml_1}$ of Δ_1'.

Since σ' is bounded from zero on the points of $\mathfrak{P}'_{-m}$ which belong to classes $\mathfrak{P}'^{ml_1}$, and since μ_m' is bounded on $\mathfrak{P}'_{-m}$, we may find a constant a_m' such that $|\mu_m'| \leqq a_m'|\sigma'|$ for all such points of $\mathfrak{P}'_{-m}$. From the property D (of the class $\mathfrak{M}''$) there is a function σ'' in $\mathfrak{M}''$ and a sequence $\{a_m''\}$ of positive constants such that, for all points of $\mathfrak{P}''$,

$$|\mu_m''| \leqq a_m''|\sigma''|.$$

Replacing $\mu_m'\mu_m''$ in inequalities (13) by $\sigma'\sigma''$ we obtain m_{em} such that $\mathrm{K}_{p_1'p_2'm_{em}}'$ implies, whenever p_1', p_2' are elements of $\mathfrak{P}'_{-m}$,

$$(14) \qquad |\Theta_{p_1} - \Theta_{p_2'}| \leqq e\,|\sigma_{p_1'}'\sigma''|.$$

For, if p_1', p_2' are connected of stage $m' \geqq m_{em}$ of Δ', they are connected of this stage in one of the two developments Δ_0', Δ_1'. In the first case $\Theta_{p_1'} = \Theta_{p_2'} = 0$ (p''), and in the second case

$$|\mu'_{mp_1'}\mu_m''| \leqq a_m'a_m''|\sigma_{p_1'}'\sigma''|.*$$

Let $\nu = \nu'\nu''$ be a common dominant of $\mu_1'\mu''$, $\mu_0\mu_1''$, $\sigma'\sigma''$. Then there is an integer m_e such that

$$(15) \qquad \mathrm{K}_{p_1'm_e}'\mathrm{K}_{p_2'm_e}'\mathrm{K}_{p_1'p_2'm_e}' \qquad \textit{implies} \qquad |\Theta_{p_1'}| + |\Theta_{p_2'}| \leqq e\,|\nu_{p_1'}|.$$

From a theorem of § 3, we have an integer m_e' (dependent on m_e) such that

* The inequality (14) can be established with less difficulty, in fact without the intervention of the system Δ_0, if the class $\mathfrak{M}'$ contains a function μ' bounded from zero on $\mathfrak{P}'_{-m}$ for every value of m. This fact led A. D. PITCHER to make this hypothesis which he introduced as the property B_3'. In the present proof the hypothesis Δ is introduced to provide functions bounded from zero on the classes $\mathfrak{P}_{-m}$. The hypothesis B_3 provides a single function effective on all $\mathfrak{P}_{-m}$. Such functions are needed in order to prove that the reduction of $\mathfrak{M}'$ relative to $\mathfrak{P}_{-m}$ has the dominance property D.

whenever $m \geqq m'_e$ then $\mathfrak{P}'_{m_e m} \leqq \mathfrak{P}'_{-m}$; and for $m = m'_e$ there is, because of inequality (14), an integer $m''_e > m'_e$ such that

$$(16) \qquad \overline{\mathrm{K}}'_{p_1 m'_e} \overline{\mathrm{K}}'_{p_2 m_e} \mathrm{K}'_{p'_1 p'_2 m''_e} \qquad \textit{implies} \qquad |\Theta_{p'_1} - \Theta_{p'_2}| \leqq e\,|\nu_{p'_1}|.$$

It follows, from the inclusion of $\mathfrak{P}'_{m_e m'_e}$ in $\mathfrak{P}'_{-m'_e}$, that $\mathrm{K}'_{p'_1 p'_2 m'_e}$ implies that both p'_1, p'_2 belong to $\mathfrak{P}_{-m'^e}$ or else neither is contained in $\mathfrak{P}'_{-m_e}$. It is now easy to see that if p'_1, p'_2 are in the relation $\mathrm{K}'_{p'_1 p'_2 m''_e}$, then one of the inequalities (15), (16) must hold. That is, there is a function μ $(= \nu)$ and positive integer m_e $(= m''_e)$ such that

$$(17) \qquad \mathrm{K}'_{p'_1 p'_2 m_e} \qquad \textit{implies} \qquad |\Theta_{p'_1} - \Theta_{p'_2}| \leqq e\,|\mu_{p'_1}|,$$

which was to be proved.

§ 6. THE PROPERTY K_{12*}.—A class $\mathfrak{M}'$ has the property K'_{12*} if for every class $\mathfrak{M}''$ with the composite property LCD* the corresponding class $(\mathfrak{M}'\mathfrak{M}'')_*$ is *equal* to the class of all functions Θ on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ which are functions of $\mathfrak{M}''$ for every element p' of $\mathfrak{P}'$ and have the property $\mathrm{K}'_{12}\mathfrak{M}'(\mathfrak{M}'')$. When the class $\mathfrak{M}''$ does not enter explicitly we drop the prime and speak of the property K_{12*} of a class $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

MOORE demonstrated that the property $\mathrm{D}\Delta\mathrm{K}_1\mathrm{K}_2$ implies the property K_{12*}. PITCHER showed that K_{12*} is implied by $\mathrm{D}_1\Delta\mathrm{K}_1\mathrm{K}_2 B_3$, where B_3 is defined in terms of the development Δ (cf. § 5). From the result of § 5 and the following theorem of MOORE, 'if $\mathfrak{M}'$ has the properties D_1, Δ', K'_1, K'_2, and $\mathfrak{M}''$ has the property LCD, every function which has the property $\mathrm{K}'_{12}\mathfrak{M}'(\mathfrak{M}'')$ and belongs to $\mathfrak{A}\mathfrak{M}''$ for every point p' of $\mathfrak{P}'$ belongs to $(\mathfrak{M}'\mathfrak{M}'')_*$,'† we have the theorem which forms the subject of this paper:

If $\mathfrak{M}$ is a class of functions on a range $\mathfrak{P}$ to the class $\mathfrak{A}$ of real numbers and has the properties D_1, Δ, K_1, K_2, *relative to a finite development Δ of $\mathfrak{P}$, then the class $\mathfrak{M}$ has the property* K_{12*}.

In view of related results of MOORE we obtain at once the further conclusion:

The class $\mathfrak{M}_*$ has the properties L, C, D_1, Δ, K_1, K_2, K_{12*}.

§ 7. ON THE INTERRELATIONS OF THE EIGHT PROPERTIES L, C, D, A, Δ, K_1, K_2, K_{12*}.—All of the eight properties, L, C, D, A, Δ, K_1, K_2, K_{12*}, have been defined previously except the property A. A class $\mathfrak{M}$ is absolute

* The property D is defined in a footnote to § 5. A class $\mathfrak{M}$ of functions is LINEAR (L) if it contains the function $a_1\mu_1 + a_2\mu_2$, whenever μ_1 and μ_2 are functions of $\mathfrak{M}$ and a_1, a_2 are real constants. A class $\mathfrak{M}$ is CLOSED (C) if it contains all functions Θ of the form

$$\Theta = \mathrm{L}_n\mu_n(\mathfrak{P};\ \mu),$$

that is, all limits of sequences of functions of $\mathfrak{M}$ converging uniformly relative to a scale function of the class $\mathfrak{M}$.

† MOORE, loc. cit., § 80, Theorem II, p. 140. The class $\mathfrak{A}\mathfrak{M}''$ is the class of all functions $a\mu''$, where a is a real number and μ'' belongs to $\mathfrak{M}''$.

(A) if it contains the absolute function $|\mu|$ of every function μ which belongs to $\mathfrak{M}$.

If a class $\mathfrak{M}$ is linear (L) and absolute (A), it has the dominance property D_1. It follows that the property $LA\Delta K_1K_2$ implies K_{12*} (§ 6). Denoting the absence of a property P by the symbol ^-P, we represent two of the three composite properties whose character was left undetermined by PITCHER as follows:

$$LC\,^-DA\Delta K_1K_2\,^-K_{12*}$$
$$L\,^-C\,^-DA\Delta K_1K_2\,^-K_{12*}.$$

It is clear that these combinations are self contradictory.

Of the 256 possible combinations of the eight properties the results of PITCHER and the present paper show that 145 exist and 110 do not exist as composite properties of a class $\mathfrak{M}$ of functions.

The remaining combination

$$LC\,^-D\,^-A\Delta K_1K_2\,^-K_{12*}$$

presents an unsolved problem. PITCHER has exhibited a number of classes of systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$ for which this composite property does not exist. From the theorem of § 6 we may add that this composite property is non-existent in any system in which the class $\mathfrak{M}$ has the property D_1.

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